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## AN ALGORITHM TO CALCULATE MINIMUM FUEL LOW THRUST TRANSFERS FOR SATELLITES AROUND THE EARTH

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# An Algorithm to Calculate Minimum Fuel Low Thrust Transfers for Satellites around the Earth

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**Abstract:** - Most of the satellite missions require orbital maneuvers to accomplish its goals. An orbital maneuver is the operation where the orbit of a satellite is changed, usually using a type of propulsion system. The maneuvers may have several purposes, such as the transfer of a satellite to its final orbit, the interception of another spacecraft or the adjustment of the orbit to compensate the shifts caused by external forces. In this situation it is essential to minimize the fuel consumption because a greater number of maneuvers can be performed, and thus the lifetime of the satellite can be extended. There are several papers and studies which aim the fuel minimization in maneuvers performed by space vehicles. In this context, this paper aims to develop and implement an algorithm capable of finding optimal trajectories with continuous thrust on a modern computing environment, which can fit different types of missions. The consideration of a realistic system with constraints, for example, the possibility to consider the limitation of the thrust direction, orbital regions where the thrust is not allowed or consider several arcs of propulsion are important characteristics of the algorithm developed here. Several maneuvers are performed for the validation of the algorithm.

**Key-Words:** - Orbital Maneuvers, Astrodynamics, Space Trajectories, Low Thrust, Sub-optimal Control.

## 1 Introduction

The problem of orbital maneuvers has many solutions available in the literature, considering different aspects of the mission and the propulsion system available. The most economical approach considers the low thrust maneuver that is studied in the present paper, where a force that has a low magnitude is applied during a certain time. To find the solutions, the equations of motion are numerically integrated. References [1] to [18] show several different aspects of this type of maneuver.

An alternative approach that is very common in the literature is based in the concept of impulsive thrust. This idea is based in a thrust that is assumed to be infinity in magnitude and instantaneous in its application. Several references show this approach, like [19] to [31].

Another idea used in orbital maneuvers is the concept of gravitational capture. References [32] to [37] work with this concept. Reference [38] explains in more detail the third-body perturbation, that is the base of the gravitational capture concept used to decrease the consumption of the orbital maneuver.

The present paper is concerned with the optimization of spacecraft trajectories with minimum fuel consumption that uses a low thrust

as a propulsion system. The main goal is a description of the algorithm developed to solve this problem that has many important features to increase the number of maneuvers that gets convergence and to increase the velocity of convergence in those situations. The problem is very sensible to small variation of parameters, so several actions are required to increase the efficiency of the convergence. The state variables and the differential equations used here are also explained.

Regarding the propulsion system, it is assumed that the thrust has a constant magnitude, its direction can be fully controlled and that it can be turned on and off at any time. In this way, to find the optimal maneuver, it is necessary to find the time to start and to stop the engine and the direction of the thrust at every instant of time, specified by the values of the pitch and yaw angles.

The conditions imposed on the final orbit must be at least in one of its Keplerian elements, but can be in more elements, depending on the goals of the mission. The algorithm also allow to consider restrictions, like forbidding thrusting in certain parts of the orbits, specified by the interval between the true longitudes. This restriction implies that both the start and the stop of the thrust have to

occur inside this interval. This is an important characteristic of the algorithm, because it is possible to avoid turning on the propulsion system when the satellite is not under direct observation from the ground. In addition, restrictions in the direction of the thrust can also be considered. This is important in order to accommodate propulsion systems that have limits in the pointing of the thrust. The algorithm can also perform maneuvers with several arcs of propulsion. These constraints and considerations make the algorithm more complex, but capable of finding optimal trajectories with realistic constraints.

## 2 Mathematical model used by the algorithm

First of all we shall introduce the notation for the orbital elements used in this paper and others important definitions. The notation is:

$a$  = semi-major axis of the orbit of spacecraft;  
 $e$  = eccentricity of the orbit of spacecraft;  
 $i$  = inclination of the orbit of spacecraft;  
 $\Omega$  = argument of the ascending node of the orbit of spacecraft;  
 $\omega$  = argument of the perigee of the orbit of spacecraft;  
 $v$  = true anomaly of the satellite;  
 $s$  = range angle of the satellite;  
 $\Phi = v + \omega - s$ .

The range angle "s" is the independent variable of this work and it is the angle between the radius position vector of the satellite and a reference arbitrary line which lies in the plane of the orbit, as shown in Fig. 1. It replaces the time in the equations of motion.

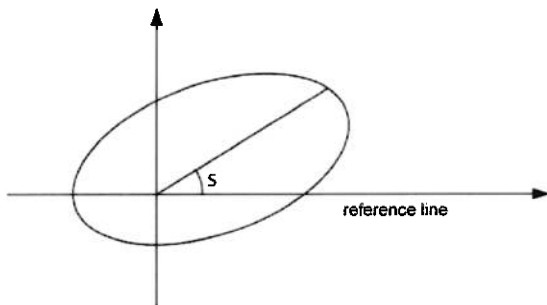


Fig.1 – Definition of the range angle and the arbitrary reference line.

There are also two important angles, pitch and yaw, which are used to define the direction of the

thrust. The angle of pitch is designated by  $A$ , while the angle yaw is designated by  $B$ . The pitch is the angle between the thrust direction and the perpendicular to the radius vector of the spacecraft, while yaw is the angle between the thrust direction and the orbital plane. The state variables used here are defined as follows (Biggs [1], Prado [3] and Oliveira [4]).

$$X_1 = \sqrt{a(1 - e^2)}/\mu$$

$$X_2 = e \cos(\omega - \Phi)$$

$$X_3 = e \sin(\omega - \Phi)$$

$$X_4 = \frac{\text{fuel consumed}}{m_0}$$

$$X_5 = \text{time} \quad (1)$$

$$X_6 = \cos\left(\frac{i}{2}\right) \cos\left(\frac{\Omega + \Phi}{2}\right)$$

$$X_7 = \sin\left(\frac{i}{2}\right) \cos\left(\frac{\Omega - \Phi}{2}\right)$$

$$X_8 = \sin\left(\frac{i}{2}\right) \sin\left(\frac{\Omega - \Phi}{2}\right)$$

$$X_9 = \cos\left(\frac{i}{2}\right) \sin\left(\frac{\Omega + \Phi}{2}\right)$$

where  $\mu$  is the gravitational constant of the Earth and  $m_0$  is the initial mass of spacecraft.

The advantage of using these state equations is to avoid singularities if the inclination or the eccentricity is zero. The equations of motion are defined as follows (Biggs [1], Prado [3] and Oliveira [4]).

$$\frac{dX_1}{ds} = f_1 = S_i X_1 F_1$$

$$\frac{dX_2}{ds} = f_2 = S_i \{ [(G_a + 1) \cos(s) + X_2] F_1 + G_a F_2 \sin(s) \} \quad (2)$$

$$\frac{dX_3}{ds} = f_3 = S_i \{ [(G_a + 1) \sin(s) + X_3] F_1 - G_a F_2 \cos(s) \}$$

$$\begin{aligned}
\frac{dX_4}{ds} &= f_4 = \frac{S_i G_a F (1 - X_4)}{X_1 W} \\
\frac{dX_5}{ds} &= f_5 = \frac{S_i G_a (1 - X_4) m_0}{X_1} \\
\frac{dX_6}{ds} &= f_6 \\
&= -\frac{S_i F_3 [X_7 \cos(s) + X_8 \sin(s)]}{2} \\
\frac{dX_7}{ds} &= f_7 = \frac{S_i F_3 [X_6 \cos(s) - X_9 \sin(s)]}{2} \\
\frac{dX_8}{ds} &= f_8 = \frac{S_i F_3 [X_9 \cos(s) + X_6 \sin(s)]}{2} \\
\frac{dX_9}{ds} &= f_9 = \frac{S_i F_3 [X_7 \sin(s) - X_8 \cos(s)]}{2}
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
G_a &= 1 + X_2 \cos(s) + X_3 \sin(s) \\
S_i &= \frac{\mu X_1^4}{[G_a^3 m_0 (1 - X_4)]} \\
F_1 &= F \cos(A) \cos(B) \\
F_2 &= F \cos(B) \sin(A) \\
F_3 &= F \sin(B)
\end{aligned} \tag{3}$$

and  $F$  is defined as the thrust magnitude and  $W$  as the exhaust velocity of the propulsion system. The control used is denoted by  $u$  and it is given by (Biggs [1], Prado [3] and Oliveira [4]):

$$\begin{aligned}
u_1 &= s_0 \\
u_2 &= (s_f - s_0) \cos(A_0) \cos(B_0) \\
u_3 &= (s_f - s_0) \sin(A_0) \cos(B_0) \\
u_4 &= (s_f - s_0) \sin(B_0) \\
u_5 &= A' \\
u_6 &= B'
\end{aligned} \tag{4}$$

The advantage of this transformation of variables is to avoid numerical problems of the type  $s_0 > s_f$ .

The true longitude, which can be used as a constraint to define the prohibit region to apply the thrust, is given by:

$$\rho = \Omega + \omega + \nu \tag{5}$$

The constraints of the final Keplerian orbital elements are given by the Equation 6. All the orbital elements are scaled. The desired values are given by the index "\*", while the suffix "0" denotes the initial value. The element without suffix or index is the current value of the element. The constraints of the final orbital elements are given by (Biggs [1], Prado [3] and Oliveira [4]):

### 3 Optimization principles used by the algorithm

The Pontryagin's maximum principle is used to find the optimal control that determinates the optimal trajectory. In this case, it is desirable to find the maximum final mass of the spacecraft, given by the functional  $J(.) = -X_4(s_f)$ . This is equivalent of minimizing the fuel consumption, because the expenditure of fuel is the only way that the satellite has to lose mass.

It is considered that  $\lambda_1, \dots, \lambda_m$  are the Lagrange multipliers associated with the constraints and that  $P_1(s), \dots, P_9(s)$  are the Lagrange multipliers associated with the differential equations of motion, usually known as adjoint multipliers.

The optimal adjoint multipliers must satisfy the adjoint equations, stated as (Biggs [2], Prado [3] and Oliveira [4]):

$$\frac{dP_i}{ds} = -\sum_{j=1}^9 P_j \frac{\partial f_j}{\partial X_i} \quad i = 1, \dots, 9 \tag{7}$$

Also, at  $s = s_f$ , the next two equations must be satisfied (Biggs [2]):

$$P_i(s_f) = -\sum_{j=1}^m \lambda_j \frac{\partial \mathcal{L}_j}{\partial X_i} \quad i = 1, \dots, 9 \text{ and } i \neq 4 \tag{8}$$

$$P_4(s_f) = -1 - \sum_{j=1}^m \lambda_j \frac{\partial \mathcal{L}_j}{\partial X_4} \tag{9}$$

where  $\mathcal{E}$  is the set of equations that defines the final orbital conditions and others constrains.

In addition, since the extreme values of the range angle  $s_0$  and  $s_f$  can be freely chosen, the next equation also must be satisfied at  $s = s_0$  and  $s = s_f$ :

$$\sum_{i=1}^9 P_i f_i = 0 \quad (10)$$

The importance of the adjoint multipliers is that, at each instant of thrusting, the optimal values of A and B can be found by solving the sub problem (Biggs [2], Prado [3] and Oliveira [4]):

$$\text{Max}_{A,B} \sum_{i=1}^9 P_i f_i \quad (11)$$

The hybrid method found in Biggs [2] is used and it simultaneously satisfies the differential equations of motion, the adjoint equations and the optimality conditions. In this way, the problem to find the optimal trajectories is a finite dimensional nonlinear programming, because the solution of the optimality condition provides the values of A and B during the maneuver.

Equation 11 can be rewrite as Equation 12 with the help of Equations 2, 8 and 9 as shown in Biggs [2], Prado [3] and Oliveira [4]:

$$\text{Max}_{A,B} \quad q_1 \cos(A) \cos(B) + q_2 \sin(A) \cos(B) + q_3 \sin(B) \quad (12)$$

where

$$\begin{aligned} q_1 &= P_1 X_1 + P_2 [X_2 + (G_a + 1) \cos(s)] \\ &\quad + P_3 [X_3 + (G_a + 1) \sin(s)], \\ q_2 &= P_2 G_a \sin(s) - P_3 G_a \cos(s), \\ q_3 &= \frac{1}{2} \{-P_6 [X_7 \cos(s) + X_8 \sin(s)] \\ &\quad + P_7 [X_6 \cos(s) - X_9 \sin(s)] \\ &\quad + P_8 [X_6 \sin(s) + X_9 \cos(s)] \\ &\quad + P_9 [X_7 \sin(s) - X_8 \cos(s)]\} \end{aligned} \quad (13)$$

Taking the derivative of Equation 12 with respect to A and B gives:

$$\begin{aligned} -q_1 \sin(A) \cos(B) + q_2 \cos(A) \cos(B) &= 0 \\ -q_1 \cos(A) \sin(B) - q_2 \sin(A) \sin(B) &+ q_3 \cos(B) = 0 \end{aligned} \quad (14)$$

From those equations and considering the fact that this is a maximization problem, the second derivative of Equation 12 must be negative. It follows, as given by Biggs [2], that

$$\begin{aligned} \sin(A) &= \frac{q_2}{E} \\ \cos(A) &= \frac{q_1}{E} \\ \sin(B) &= \frac{q_3}{S} \\ \cos(B) &= \frac{E}{S} \end{aligned} \quad (15)$$

$$\begin{aligned} \text{where } E &= \pm \sqrt{q_1^2 + q_2^2} \\ S &= \pm \sqrt{q_1^2 + q_2^2 + q_3^2}. \end{aligned} \quad \text{and}$$

Using this approach, it's only necessary to give the initial conditions and then the direction of the optimal angles are given by the solution of Equation 11 at each step of the integration of the equations of motion. The formulation of the problem is given by the Equations 16 to 18.

$$\text{Maximize:} \quad -X_4(s_f) \quad (16)$$

$$\text{With respect to:} \quad s_0, s_f, A_0, B_0, A'_0, B'_0, P_i(s_0) \quad (17)$$

$$\text{Subjected to:} \quad S_j(X(s_f)) = 0, \quad j = 1, \dots, m \quad (18)$$

where  $S_j(X(s_f))$  are the constraints stated in Equations 5 and 6.

In this way, a sketch of the algorithm is given by:

- a) An initial set of the variables  $s_0, s_f, A_0, B_0, A'_0, B'_0, P_i(s_0)$  are guessed;

- b) A numerical integration of the adjoint equations together with the equations of motion are performed simultaneously, from the range angle  $s_0$  until a guessed final value of the range angle  $s_f$ . At every step of integration the values of  $A$  and  $B$ , are obtained from Equation 11;
- c) At this point the satisfaction of the constraints is verified, as well as the fuel minimization. If one of the conditions is not satisfied, according to a specified tolerance, a gradient method is applied to find a new set of initial variables and the algorithm goes back to step "a". If the tolerances are satisfied, the procedure finishes.

#### 4 Adjoint-Control Transformation

From the last part of the present paper, one of the initial variables that should be guessed are the initial adjoint multipliers  $P_i(s_0)$ . A good initial guess for those variables is very important to assist convergence and to reduce the time the algorithm needs to obtain the solution.

Unfortunately, the Lagrange multipliers do not have any physical meaning, so it is hard to find a good initial guess for them. To solve this problem, the adjoint-control transformation is used to attribute a good estimative for  $P_i(s_0)$  from variables that have physical meaning. Those variables are the angles  $A_0$  and  $B_0$  and also the values of  $A'_0$  and  $B'_0$  (Prado [3], Biggs [2]). The justification for this change from nine variables to four comes from Biggs [2] and Prado [3]:

- a) The Lagrange multipliers are not all independent. Except for the state variables  $X_4$  and  $X_5$ , the last seven variables are a linear combination of the five Keplerian elements. So, two Lagrange multipliers are dependent variables and the independent Lagrange multipliers can be reduced to seven.
- b) The adjoint equations are homogeneous, so the same trajectory can be found by using initial values of Lagrange multipliers given by  $P_i(s_0)$  or  $kP_i(s_0)$ , where  $i = 1, \dots, 9$  and  $k$  is a positive constant. So, it is possible to normalize the Lagrange multipliers, reducing in one unit the number of the independent components of  $P_i$ , diminishing this number to six.

- c) The trajectories are time independent, so  $P_5$  will always be zero, because  $\frac{\partial P_5}{\partial s} = 0$  and  $\frac{\partial s_f}{\partial x_5} = 0$ . So, there is a new reduction in one unit in the number of independent components of  $P_i$ .
- d) The range angle of the beginning of the maneuver is free, so at  $s = s_0$  Equation 10 is satisfied, reducing finally the number of independent variables to four.

Reducing from nine to four the number of the independent Lagrange multipliers, now it's possible to apply the adjoint-control transformation to find a good estimative for the initial Lagrange multipliers using variables with physical meaning. This task can be performed by guessing the variables  $A_0$ ,  $B_0$ ,  $A'_0$  and  $B'_0$ , because the values are  $A_0$  and  $B_0$  at  $s = s_0$  and  $A_0 + 5A'_0$  and  $B_0 + 5B'_0$  at  $s = \tilde{s}$  ( $\tilde{s} = s_0 + 5^\circ$ ), where the value of 5 degrees is chosen arbitrarily, can be replaced in Equation 11, to obtain the initial Lagrange multipliers. It is important to note that, during the interval  $s_0$  and  $\tilde{s}$ , the propulsion is turned off, so  $P_i(s_0) = P_i(\tilde{s})$  as stated in Biggs [2].

This transformation provides means of finding a reasonable estimative for the values for  $P_i(s_0)$  with variables with physical meaning, so providing good numerical stability and faster convergence for the algorithm proposed here.

#### 5 Steps of the Algorithm

The final complete algorithm has the following steps (Prado [3] and Oliveira [4]):

- I. It starts from an initial guess for the set of variables ( $A_0$ ,  $B_0$ ,  $A'_0$ ,  $B'_0$ ,  $s_0$ ,  $s_f$ ). With these initial values, the initial guess for the control vector  $u$  is obtained from Equation 4;
- II. The "adjoint-control" transformation is used to obtain the initial values of the Lagrange multipliers required for the numerical integrations, as described before in section 4;
- III. Now, with all the initial values needed to solve the problem, the equations of motion and the adjoint equations are simultaneously numerically integrated during the propulsion arc. The values of the pitch and yaw angles are obtained, at every step of the integration, by the solution of Equation 11 that represents the Principle of Maximum of Pontriagyn. By the

end, the value of the final state  $X$  after the propulsion arc is obtained;

- IV. The Keplerian elements of the orbit achieved are then obtained from the final state  $X$  using Equations 1;
- V. At this point, the satisfaction of the constraints is verified. If the magnitude of the vector that represents the constraints satisfaction is smaller than a specified tolerance provided by the user (the numerical zero), the algorithm proceeds to step vii. On the other hand, if it is bigger, the algorithm proceeds to the step vi;
- VI. At this step, the gradient of the constraints equations ( $\nabla S$ ) is obtained by perturbing the elements of the control vector  $u$  and performing a new set numerical integrations, in order to obtain the numerical derivatives of the constraints equations with respect to the vector  $u$ . Then, a new set of values for the vector  $u$  is found by the equation (Luemberger [39]):

$$u_{i+1} = u_i - \nabla S^T [\nabla S \nabla S^T]^{-1} \nabla S \quad (19)$$

and then the algorithm goes to step i, with the new value of the vector  $u$ ;

- VII. Once the constraints satisfaction is achieved, the next step is to search for the minimum of the fuel consumed. The direction of the search  $d$ , using the gradient method (Bazarra [40]), is;

$$d = -R \nabla J(u) \quad (20)$$

where the vector  $R$  is

$$R = I - \nabla S^T [\nabla S \nabla S^T]^{-1} \nabla S \quad (21)$$

- VIII. At this step, the magnitude of the vector  $d$  is verified. If it is smaller than a value specified by the user as a direction search tolerance (numerical zero), the algorithm goes to step x and, if it is bigger, the algorithm proceed to step ix.
- IX. In order to get better results, the initial data "ratio of contraction"  $RC$  and the direction search tolerance is reduced as  $u$  gets closer to

the minimum. The magnitude for the direction search is then given by:

$$PB = \frac{RC J(u)}{\nabla J(u) d} \quad (22)$$

So, the complete step is given by:

$$u_{i+1} = u_i + PB \frac{d}{|d|}, \quad (23)$$

and then algorithm goes back to step i;

- X. At this step, the possibility of the control  $u$  be a Kuhn-Tucker point is verified. To check this condition, the vector  $VT$  is build, formed by the  $m$  first elements of the vector given by (Bazarra 40]]:

$$W = -(SS^T)^{-1} S \nabla J(u) \quad (24)$$

where  $m$  is the number of active constraints. If  $VT$  is not positive definite, the line  $j$ , which provides  $VT_j$  to be negative, of the matrix  $S$  is deleted, and then the algorithm returns to step v.

- XI. The current step is verified and, if it is not the last one, the algorithm goes to a new search for the minimum fuel consumption with a smaller direction search tolerance for the objective function. If this step is the last one, the algorithm is finalized.

## 6 Extensions of the algorithm proposed

There are several extensions of the algorithm that can be considered, in order to make it more flexible.

The first one is the possibility to perform maneuvers with more than one arc of propulsion. Increasing the number of propulsion arcs means that the number of optimization variables is increased. For example, for one thrust arc there are  $u_1$  to  $u_6$  control variables. For two thrust arcs, similarly, there is a set of  $u_1$  to  $u_{12}$  control variables and so on. In addition, adding more than one thrust arc, the inclusion of constraints to prevent the overlap of thrust arcs becomes necessary.

Another extension is the possibility to consider forbidden regions to apply the propulsion as a constraint. These regions are defined by the true longitude, as shown in Equation 5. Both values of the initial and final true longitudes of the prohibit region have to be specified.

In some situations, the initial range angles  $s_0$  can be less than zero. It means that when the thrust is started, the time is negative. To avoid negative times, it's added a constraint that  $s_0$  should be equal or bigger than zero. This constraint is only active if  $s_0 < 0$ .

The last extension is the possibility to consider constraints restricting the pitch and yaw angles. In this case the bound considered on  $A$  and  $B$  are given by Biggs [2]:

$$A_u \geq A \geq A_L \quad (25)$$

$$B_u \geq B \geq B_L \quad (26)$$

where the suffix "u" represents the maximum angle and the suffix "L" represents the minimum angle.

This constraint is active when Equations 25 and 26 are false after finding the optimal angles  $A^*$  and  $B^*$  and this problem is solved as follows (Biggs [2]):

$$A = A_u \text{ if } A^* \geq A_u \quad (27)$$

$$A = A_L \text{ if } A^* \leq A_L \quad (28)$$

$$A = A^* \text{ se } A_u \geq A^* \leq A_L \quad (29)$$

Equations 27 to 29 are similarly applied to the optimal yaw angles  $B^*$ .

## 7 Results

In order to test the algorithm developed, some optimal trajectories were found. The maneuvers available in Biggs, [1, 2] and Prado, [3] were chosen. In this way, it's possible to validate the algorithm and to be able to use it in many different kinds of maneuvers and purposes. It should be noted that the most important applications of the algorithm proposed are in situations where several orbital elements of the final orbit are specified. Several of the examples shown here consider more

simple situations, where only the semi-major axis needs to be changed. For those situations, simpler solutions are available in the literature. Anyway, those situations are considered here to allow comparisons with Biggs [1, 2] and Prado [3]. In particular, Biggs [1, 2] have several maneuvers simulated, but no algorithm is presented to obtain the solutions. To make a more complete study, after making those comparisons, the same maneuvers used are extended to consider more propulsion arcs, limits in the control, etc. Those extensions are made in order to verify the extra costs involved and to test those capabilities of the algorithm proposed here in situations where no simpler solutions are available.

### 7.1 Maneuver 1

Initial orbital elements:

$a = 99000 \text{ km}$ ;  $e = 0.7$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  $v = -105^\circ$

Vehicle mass = 300 kg; Thrust = 1.0 Newton;

Condition imposed on the final orbit:  $a = 104000 \text{ km}$ ;

Solution:

Final orbital elements:

$a = 103999.72 \text{ km}$ ;  $e = 0.714359$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  $v = 28.2426^\circ$

Fuel consumption: 2.4197 kg; Fuel consumption in Biggs [2]: 2.4393 kg; Fuel consumption in Prado [3]: 2.4390 kg;

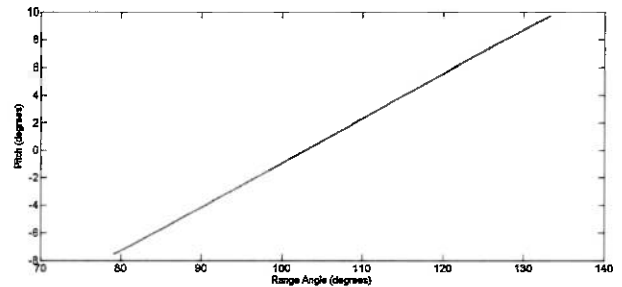


Fig.2 - Optimal pitch angles of the maneuver 1

It's possible to view that the fuel consumption of the algorithm developed is better than the fuel consumed in the references.

## 7.2 Maneuver 2

This maneuver has the same initial and final conditions of the previous example with now two thrust arcs instead of one.

Initial orbital elements:

$a = 99000 \text{ km}$ ;  $e = 0.7$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  
 $\nu = -105^\circ$

Vehicle mass = 300 kg; Thrust = 1.0 Newton,  
Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 104000 \text{ km}$ ;

Other constraints: None.

Final orbital elements after the first propulsion arc:

$a = 102036.92 \text{ km}$ ;  $e = 0.7089$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  
 $\omega = 105^\circ$ ;  $\nu = 16.0355^\circ$

Final orbital elements after the second propulsion arc:

$a = 103999.89 \text{ km}$ ;  $e = 0.71440$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  
 $\omega = 105^\circ$ ;  $\nu = 9.0578^\circ$

Fuel consumption: 2.4076 kg;

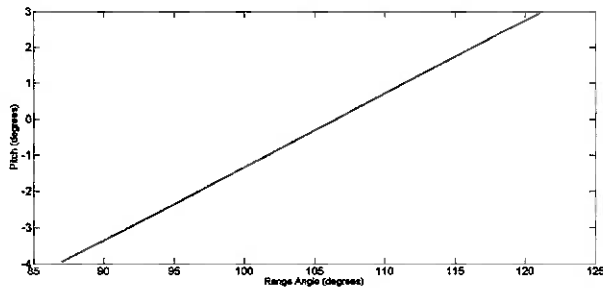


Fig.3 - Optimal pitch angles of the maneuver 2 for the first arc

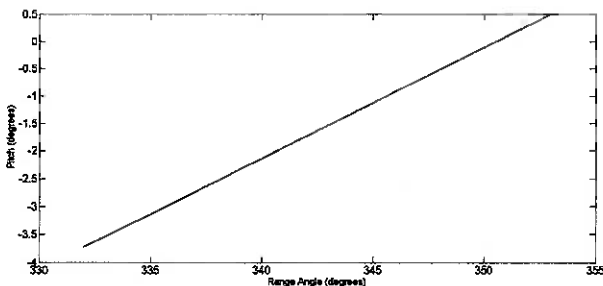


Fig.4 - Optimal pitch angles of the maneuver 2 for the second arc

This maneuver with two arcs of propulsion had the fuel consumption reduced. This reduction was expected since that with two arcs of propulsion there are more parameters to optimize. The reduction of the fuel consumption compared to the maneuver 1 was 0.0121 kg.

## 7.3 Maneuver 3

This maneuver has the same initial and final conditions as the previous examples with three thrust arcs.

Initial orbital elements:

$a = 99000 \text{ km}$ ;  $e = 0.7$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  
 $\nu = -105^\circ$

Vehicle mass = 300 kg; Thrust = 1.0 Newton,  
Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 104000 \text{ km}$ ;

Other constraints: None.

Final orbital elements after the first propulsion arc:

$a = 101623.95 \text{ km}$ ;  $e = 0.7077$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  
 $\omega = 105^\circ$ ;  $\nu = 16.1995^\circ$

Final orbital elements after the second propulsion arc:

$a = 103101.90 \text{ km}$ ;  $e = 0.7119$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  
 $\omega = 105^\circ$ ;  $\nu = 8.6222^\circ$

Final orbital elements after the third propulsion arc:

$a = 103999.37 \text{ km}$ ;  $e = 0.71441$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  
 $\omega = 105^\circ$ ;  $\nu = 5.4578^\circ$

Fuel consumption: 2.4061 kg;

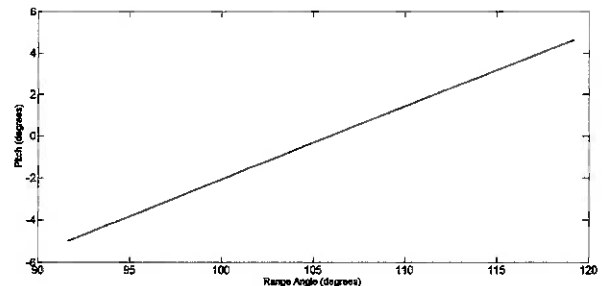


Fig.5 - Optimal pitch angles of the maneuver 3 for the first arc

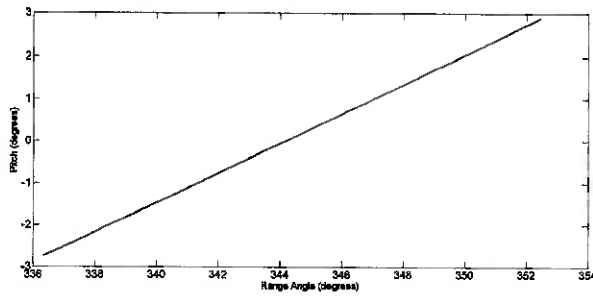


Fig.6 - Optimal pitch angles of the maneuver 3 for the second arc

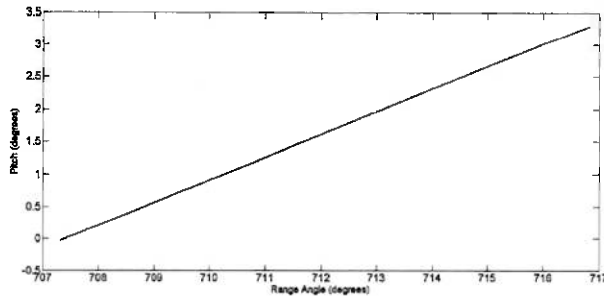


Fig.7 - Optimal pitch angles of the maneuver 3 for the third arc

This maneuver now with three arcs of propulsion had the fuel consumption reduced compared to the same maneuver with two arcs. This reduction was expected as the parameters to optimize were increased. The reduction of the fuel consumption compared to the maneuver 2 was  $1.5 \times 10^{-3}$  kg. This tendency of the consumption of stabilizing in a fixed value is expected, since there is a limit in the possible reduction due to the increase in the number of propulsion arcs.

#### 7.4 Maneuver 4

This maneuver has the same initial and final condition as the previous examples with one thrust arc and an additional constraint at the pitch angle. The idea is to test this capability of the proposed algorithm.

Initial orbital elements:

$a = 99000$  km;  $e = 0.7$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  $\nu = -105^\circ$

Vehicle mass = 300 kg; Thrust = 1.0 Newton, Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 104000$  km;

Other constraints:  $5^\circ \geq A \geq -5^\circ$  (Pitch constraint).

Final orbital elements:

$a = 103999.21$  km;  $e = 0.7143$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  $\nu = 30.24^\circ$

Fuel consumption: 2.4235 kg; Fuel consumption in Biggs [2]: 2.44 kg; Fuel consumption in Prado [3]: 2.45 kg;

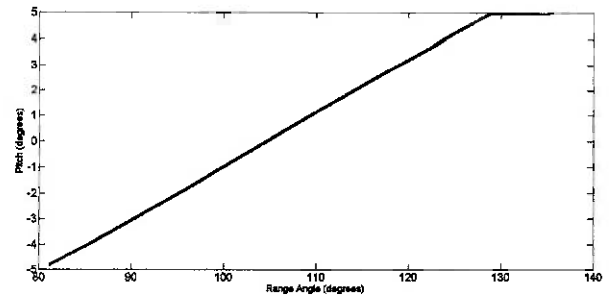


Fig.8 - Optimal pitch angles of the maneuver 4

Now with the constraint in the direction of propulsion the fuel consumption was increased. The extra fuel needed in this maneuver compared to the maneuver 1 was  $3.8 \times 10^{-3}$  kg. It's possible to view clearly that the pitch angle is confined in  $-5^\circ$  to  $5^\circ$ . In any case, the algorithm proposed here obtain better results when compared to Biggs [2] and Prado [3].

#### 7.5 Maneuver 5

This maneuver has the same initial and final condition as the previous examples with one thrust arc and an additional constraint of a region prohibited for thrusting. Once again the idea is to test this capability of the algorithm shown here.

Initial orbital elements:

$a = 99000$  km;  $e = 0.7$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  $\omega = 105^\circ$ ;  $\nu = -105^\circ$

Vehicle mass = 300 kg; Thrust = 1.0 Newton, Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 104000$  km;

Other constraints: No thrust between the true longitudes  $120^\circ$  and  $180^\circ$ .

Final orbital elements:

$a = 103999.85$  km;  $e = 0.7134$ ;  $i = 10^\circ$ ;  $\Omega = 55^\circ$ ;  
 $\omega = 105^\circ$ ;  $v = 320.80^\circ$

Fuel consumption: 2.7852 kg; Fuel consumption in Biggs [2]: 2.81 kg; Fuel consumption in Prado [3]: 2.81 kg;

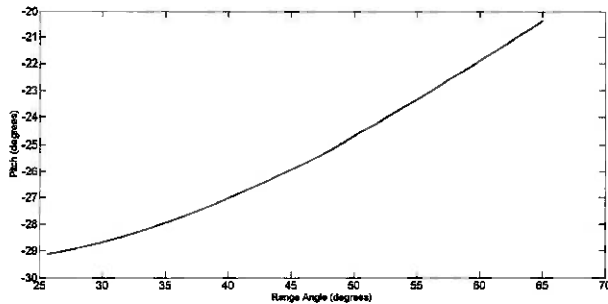


Fig.9 - Optimal pitch angles of the maneuver 5

In figure 7 it's possible to note that the solution has the thrust arc ending in  $\lambda = 120^\circ$ . In this way, the region prohibit of thrust was obeyed and, compared with maneuver 1, the extra fuel needed to reach the semi-major axis constraint was 0.3655 kg. Once again, the algorithm proposed here obtain better results when compared to Biggs [2] and Prado [3].

## 7.6 Maneuver 6

Initial orbital elements:

$a = 9900$  km;  $e = 0.2$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  $\omega = 25^\circ$ ;  $v = -10^\circ$

Vehicle mass = 300 kg; Thrust = 2.0 Newton,  
 Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 10000$  km;

Final orbital elements:

$a = 9999.9180$  km;  $e = 0.2054$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  
 $\omega = 24.9^\circ$ ;  $v = 95.09^\circ$

Fuel consumption: 3.2897 kg; Fuel consumption in Biggs [2]: 3.7222 kg; Fuel consumption in Prado [3]: 3.7316 kg;

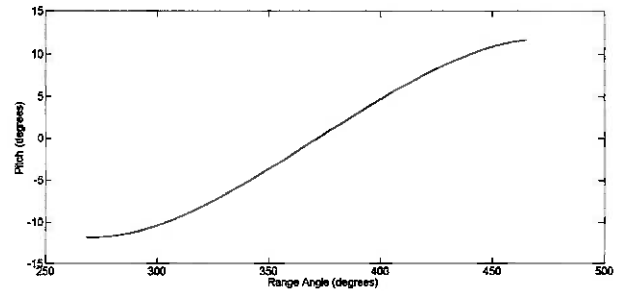


Fig.10 - Optimal pitch angles of the maneuver 6

It's possible to view that the fuel consumption of the developed algorithm was a lot better than the fuel in the references for this maneuver since the fuel consumption was more than 0.56 kg less than the references. This large savings in very interesting and shows that all the research made in the cited references are searching for local minimums. In this example, the algorithm proposed here was able to find a better local minimum with respect to the ones found in Biggs [2] and Prado [3].

## 7.7 Maneuver 7

This maneuver has the same initial and final condition as the maneuver 6, but now with two thrusting arcs.

Initial orbital elements:

$a = 9900$  km;  $e = 0.2$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  $\omega = 25^\circ$ ;  $v = -10^\circ$

Vehicle mass = 300 kg; Thrust = 2.0 Newton,  
 Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 10000$  km;

Final orbital elements after the first thrusting arc:

$a = 9938.8684$  km;  $e = 0.2019$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  
 $\omega = 24.73^\circ$ ;  $v = -6.23^\circ$

Final orbital elements after the second thrusting arc:

$a = 9999.9775$  km;  $e = 0.2057$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  
 $\omega = 24.33^\circ$ ;  $v = 33.59^\circ$

Fuel consumption: 3.2750 kg;

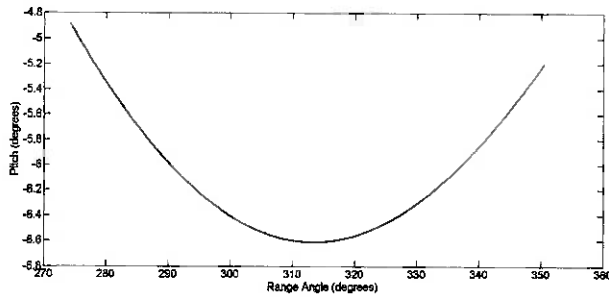


Fig.11 - Optimal pitch angles of the maneuver 7 for the first arc

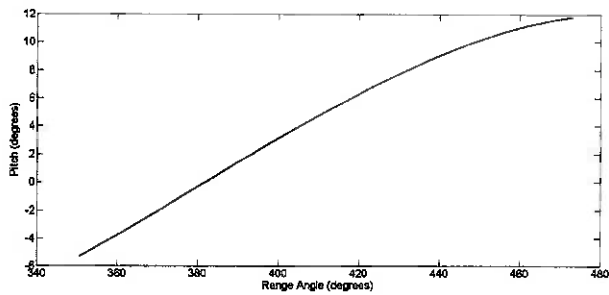


Fig.12 - Optimal pitch angles of the maneuver 7 for the second arc

This maneuver with two arcs of propulsion had the fuel consumption reduced by 0.0147 kg compared to the maneuver 6. This reduction was expected, since that there are more parameters to optimize with two arcs of propulsion.

## 7.8 Maneuver 8

This maneuver has the same initial and final condition as the previous maneuvers 6 and 7 with three thrust arcs.

Initial orbital elements:

$a = 9900$  km;  $e = 0.2$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  $\omega = 25^\circ$ ;  $v = -10^\circ$

Vehicle mass = 300 kg; Thrust = 2.0 Newton,  
Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 10000$  km;

Final orbital elements after the first thrusting arc:

$a = 9936.4577$  km;  $e = 0.2027$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  
 $\omega = 25.10$ ;  $v = 49.1836^\circ$

Final orbital elements after the second thrusting arc:

$a = 9969.8371$  km;  $e = 0.2052$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  
 $\omega = 25.24^\circ$ ;  $v = 50.14^\circ$

Final orbital elements after the third thrusting arc:

$a = 10000.04$  km;  $e = 0.2075$ ;  $i = 10^\circ$ ;  $\Omega = 0^\circ$ ;  $\omega = 25.38^\circ$ ;  $v = 47.49^\circ$

Fuel consumption: 3.1233 kg;

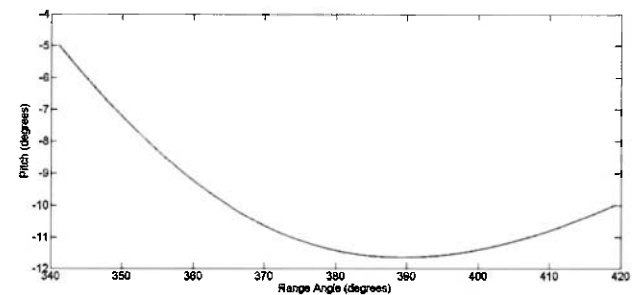


Fig.13 - Optimal pitch angles of the maneuver 8 for the first arc

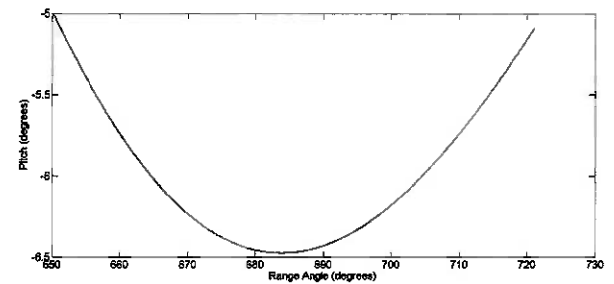


Fig.14 - Optimal pitch angles of the maneuver 8 for the second arc

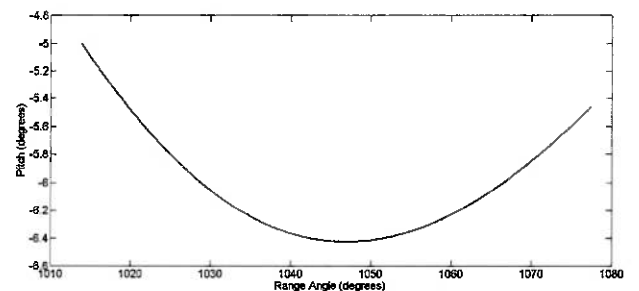


Fig.15 - Optimal pitch angles of the maneuver 8 for the third arc

This maneuver with three arcs of propulsion had the fuel consumption reduced by 0.1517 kg compared to the maneuver 7. This reduction was expected, since that there are more parameters to optimize with three arcs of propulsion.

## 7.9 Maneuver 9

This maneuver has the requirements of changing three orbital elements of the orbit at the same time. It is the more interesting and difficult situation.

Initial orbital elements:

$a = 41904.1$  km;  $e = 0.018$ ;  $i = 0.688^\circ$ ;  $\Omega = -$

$29.8^\circ$ ;  $\omega = 7^\circ$ ;  $\nu = -97.2^\circ$

Vehicle mass = 300 kg; Thrust = 1.0 Newton,  
Initial position  $s = 0^\circ$ ;

Condition on the final orbit:  $a = 42164.2$  km;  $e = 0$ ;

$i = 0^\circ$ ;

Final orbital elements:

$a = 42164.55$  km;  $e = 0.00130$ ;  $i = 0.0419^\circ$ ;  $\Omega = 300.60^\circ$ ;  $\omega = 100.47^\circ$ ;  $\nu = 143.64^\circ$

Fuel consumption: 5.565 kg; Fuel consumption in Biggs [2]: 5.621 kg; Fuel consumption in Prado [3]: 5.579 kg;

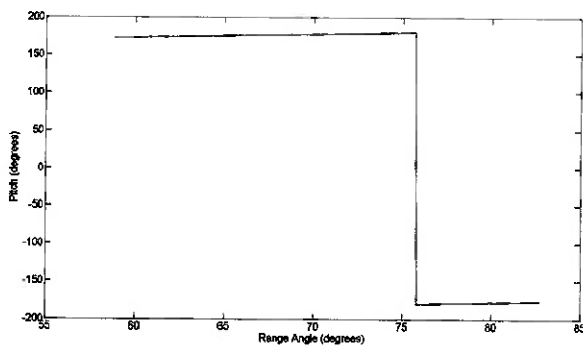


Fig.16 - Optimal pitch angles of the maneuver 9 for the first arc

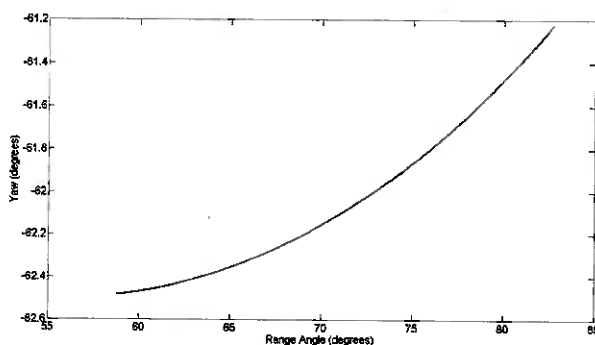


Fig.17 - Optimal yaw angles of the maneuver 9 for the first arc

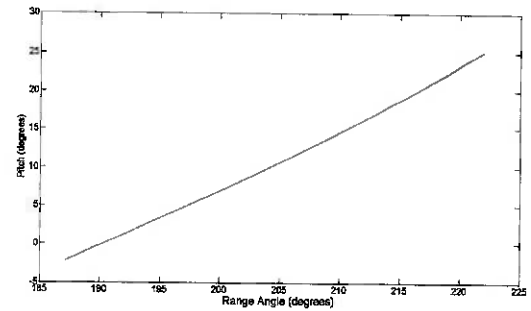


Fig.18 - Optimal pitch angles of the maneuver 9 for the second arc

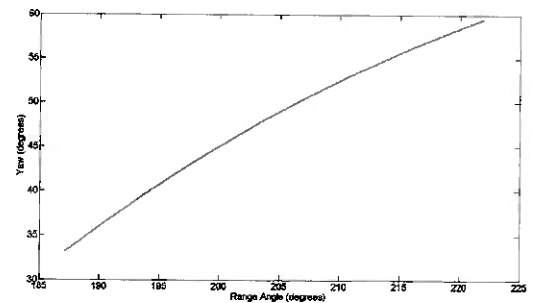


Fig.19 - Optimal yaw angles of the maneuver 9 for the second arc

In this maneuver there are three keplerian elements conditions for the final orbit, the semi-major axis, the eccentricity and the inclination. Since the maneuver includes a change in the initial orbit plane, the yaw angle appears to provide it.

The final conditions were satisfied and the fuel consumption was reduced by 0.014 kg compared to Prado [3].

## 8. Conclusion

This paper proposed an algorithm to solve the problem of orbital maneuvers using a low thrust control for a spacecraft that is travelling around the Earth.

All the maneuvers studied in this paper were successfully solved for different final conditions and constraints with reasonable fuel consumptions.

The fuel consumptions obtained by the algorithm proposed here were always lower than those of the references. In this way, the algorithm showed a good efficiency to find optimal trajectories.

It's possible to note that maneuvers with the same initial and final conditions have lower fuel consumption as the number of thrusting arcs is increased. This occurs because the algorithm has more variables to optimize as the number of arcs are increased, minimizing the fuel consumption.

Maneuvers with constraint in the direction of the thrust or prohibited thrust regions have a higher fuel consumption compared to maneuvers without those constraints.

It's important to point out that the algorithm developed does not guarantee that it will find a global minimum for the problem. The solution is always a local minimum, since it's considered only the first order conditions to find the optimal maneuver. And also, the solution of the algorithm developed can be dependent on the initial guessed solution.

In conclusion, the algorithm has successfully found optimal maneuvers for a number of different maneuvers found in the literature, using different constraints. The algorithm was validated and now it's possible to use it for many kinds of mission, using both low or high thrust propulsion and with different constraints.

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