

## Solution for anomalous diffusion equation with source term

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**Abstract:** In literature the phenomenon of diffusion has been widely studied, however for non-extensive systems which are governed by a stochastic nonlinear dynamics, there are few soluble models. The purpose of this paper is to present the solution of the non-linear Fokker-Planck equation for a model of potential with barrier considering a term of absorption. Systems of this nature can be observed in various chemical or biological process and its solution enriches the studies of non-extensive systems exist.

**keywords:** Stochastic Dynamics, Nonextensive systems, Applications of Nonlinear Sciences.

### 1. BASIC INFORMATION

A large number of phenomena can be related to kinetic processes, this is the case, for example, electrochemical reactions in metals or in membranes. In these cases, the atoms or particles that are interacting at the interface between the environments, bulk and the interior of the material. This difference between the environments in many cases leads to a diffusion process in which particles of the medium are absorbed into the structure. This absorption can occur at a constant rate or in some cases, has time dependence [1].

The Fokker-Planck equation is widely used in literature to study stochastic processes of diffusion, where one can consider the presence of forces acting on the interaction between the bodies [2]. In the problem addressed, we will make a brief review of nonlinear diffusion in a simple system consisting of two regions separated by a barrier of potential.

For the study, we adopt the non-linear Fokker-Planck equation (NLFP) with an additional term time, which is interpreted as a source term or absorption that favors the passage of particles through the potential barrier.

### 2. PURPOSE

The non-linear Fokker-Planck equation can be write as follows [3]:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} [P(x,t)f(x)] + Q \frac{\partial^2 P(x,t)^\nu}{\partial x^2} \quad (1)$$

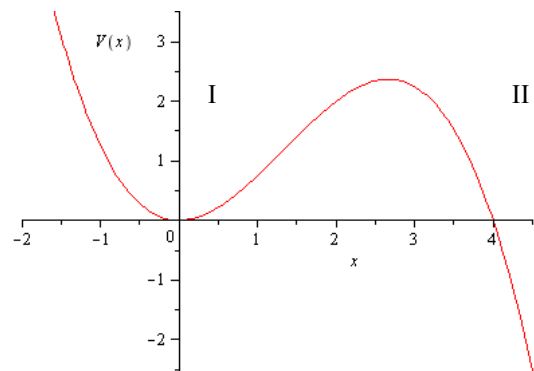
In this equation,  $f(x)$  represents a force of interaction among the particles of the external environment with the

internal and  $P(x,t)$  can be considered the density of particles of a medium.

An additional term, proportional to the function  $\mu(t)$  is included in equation (1). This term is related to the a source of the model. Thus, we obtain:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} [P(x,t)f(x)] + Q \frac{\partial^2 P(x,t)^\nu}{\partial x^2} - \mu(t)P(x,t) \quad (2)$$

The nonlinearity of the Fokker-Planck equation is due to the exponent  $\nu$  present in second derivative. Equation (1) is employed in the study of non-extensive systems, where there are long-range interactions and the thermodynamic relations serve the proposal introduced by Tsallis [4-5].



**Figure 1: Representation of the potential considered with  $k_1 = 1$  and  $k_2 = 0.25$ .**

In this paper, the function  $f(x)$  is given by derivative of potential:

$$V(x) = k_1 x^2 - k_2 x^3, \quad (3)$$

whose characteristic graph is shown in Figure 1. The solution for this system is of the form:

$$P(x,t) = D(t) \cdot \{1 - (\nu - 1) \cdot \beta(t) \cdot (x^2 - \zeta(t) \cdot x^3)\}^{1/(\nu-1)} \quad (4)$$

where  $D(t)$ ,  $\beta(t)$  and  $\zeta(t)$  are dependent functions of the time to be defined later. Initially, for a time equal to zero, the *Ansatz* above gives the distribution of the stationary Fokker-Planck equation. As we are working in non

extensive, the solution needs to be given this way, thus maximizing the Tsallis entropy and thermodynamics allows us to draw relevant information from the system in future work.

### 3. METHODS AND RESULTS

The method applied to obtain the solution of expression (2) was a substitute the *Ansatz* (4) in NLFP. This substitution provides a system of equations that defines the temporal functions and source term to satisfy the problem.

The expressions obtained for  $D(t)$ ,  $\beta(t)$  and  $\zeta(t)$  was:

$$D(t) = \left\{ \frac{Qv}{k_1} + \left( 1 - \frac{Qv}{k_1} \right) e^{2k_1 t} \right\}^{-2/(v-1)}, \quad (5)$$

$$\beta(t) = \left\{ \frac{Qv}{k_1} + \left( 1 - \frac{Qv}{k_1} \right) e^{2k_1 t} \right\}^2, \quad (6)$$

$$\zeta(t) = \left\{ \frac{Qv}{k_1} + \left( 1 - \frac{Qv}{k_1} \right) e^{2k_1 t} \right\} \quad (7)$$

And the source term get to this model is:

$$\mu(t) = 2Qv \frac{(v+2)}{(v-2)} \left\{ \frac{Qv}{k_1} e^{-2k_1 t} - 1 \right\}^{-1} \quad (8)$$

where  $Q$  is the diffusion constant and  $v$  the parameter of nonlinearity of FPNL.

In the Figure 2 presents the graph of the distribution  $P(x, t)$  obtained in a time interval equal to 2 units, with  $x$  ranging from -1.5 to 2, which corresponds to the region of stable potential minimum (region I). The values assigned to constants were:  $k_1 = 1$ ,  $k_2 = 0.25$ ,  $v = 1.5$  e  $Q = 0.5$ .

The time for the particle density to go the zero, with these values of the constants was of approximately 1 unit.

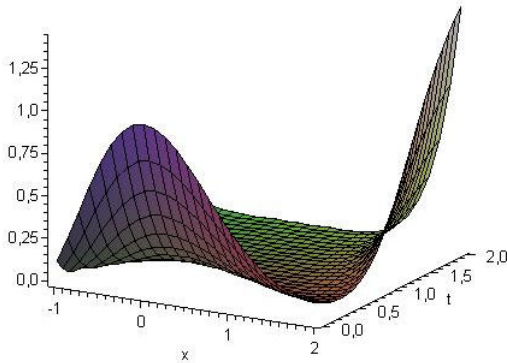


Figure 2: Density of particle by space and time.

### 4. DISCUSSION

The inclusion of a term referring to an absorption rate has been treated in some papers (see references [6] and [7]). However, in these cases, the function  $f(x)$  corresponds to a harmonic potential.

In this model, was inserted into a disturbance term in the harmonic potential barrier and creating two-state levels. This change has generated a time dependence specific for this case in the source term (8).

Temporal dependencies of the kind presented here (8) can be observed in many biochemical systems [8-9] when analyses rate of absorption that open the possibility

of approximate this theoretical model with data of reactions or interactions obtained in the laboratory. The limitation in these cases is the need to adjust the parameter non-linear ( $v$ ), as the system being treated.

### 5. CONCLUSION

The graphics made from the *Ansatz* suggested show an expected behavior of the system, i.e., as time passes there is a tendency for particles to go from a region of higher potential for a minor.

This diffusion process is favored not only by a factor of nonlinear Fokker-Planck equation, but also by adding the source term  $\mu(t)$  that helps the particles to transpose the barrier. The incorporation of this function in temporal Fokker-Planck equation nonlinear led to his solution to a simple system of the potential barrier.

In literature there are few models for soluble FPNL, which hinders the application of non-extensive formalism in the study of chemical or biological process.

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