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TRANSIENT DYNAMICS AND SYNCHRONIZATION REGIONS OF PULSE-COUPLED PIECEWISE LINEAR OSCILLATORS IN T^2

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Abstract: Synchronization times and stable states of pulse-coupled Light-Controlled Oscillators (LCOs) are obtained for different coupling configurations and intensities as a function of the initial conditions. Transients detection methods are compared in terms of their efficiency to detect synchronization times, emerging the phase return-map as the best candidate. Scaling laws that restric the transient dynamics are obtained, based, on dimensional considerations. A qualitative study of different oscillatory regimes and the topology of the orbits is performed.

keywords: synchronization, relaxation oscillators, coupling configurations

Synchronization is a common feature of oscillatory systems and might be understood as an adjustment of rhythms of self-sustained oscillators due to weak interactions. It constitutes an ubiquitous phenomenon and nowadays is a widely spread topic which several books have been devoted to, both from rigorous [1] to popularization point of view [2]. Synchronous regimes of arbitrary order [1] are observed when studying coupled oscillators, and the whole family of synchronization regions, called Arnold tongues, might be plotted involving coupling strengths between the oscillators and their frequencies or periods.

The study of how coupled oscillators achieve synchrony is important due to its relevance to experimental observations of synchronous neural firing patterns of various mammalians, insects and reptilian species. In particular, relaxation oscillators and coupling configurations are specially relevant for these studies. Here we consider the synchronization times and stable states of Light-Controlled Oscillators (LCOs), which constitute unidimensional relaxation oscillators described by two distinct time scales meant to mimic fireflies in a simple fashion [3, 4].

Each LCO is composed of an LM555 chip with an astable oscillating mode and a dual RC circuit that shares the capacitor. The chip element chooses the corresponding RC by establishing well-defined thresholds for the charging and discharging states at $2V_M/3$ and at $V_M/3$ respectively, where V_M is the source voltage value.

LCOs can interact with different strengths by means of IR transmitter diodes and, as a result of the interaction, their natural periods are modified. The configurations dealt in this presentation are the master-slave (MS) and the mutual-interaction (MI). The LCOs are described by the following set of autonomous differential equations [3, 4]:

$$\dot{V}_i(t) = \lambda_i \left[V_M - V_i(t)\right] \epsilon_i(t) - \gamma_i V_i(t) \left[1 - \epsilon_i(t)\right] + \sum_{j=1,\, j\neq i}^{N} \beta_{ij} \left[1 - \epsilon_j(t)\right],\ i = 1, \ldots, N, \qquad (1)$$

where V_i is the i-th LCO voltage, β_{ij} gives account of the coupling, and $\epsilon_i(t)$ represents the oscillator stage that takes the value 1 (charging stage) or 0 (discharging stage). The parameter λ_i (γ_i) is the inverse characteristic time scale for the charging (discharging) stage.

In the case of an isolated LCO the dynamics is naturally represented in the circle S^1. This state space has an injective and a dissipative part of similar length. Through the injective part, the LCO charges during a time T_λ, while being in the dissipative part a fast discharge occurs lasting T_γ. Every time in any LCO achieves a threshold a new initial condition for the global system is generated. The action of the coupling is to raise the asymptotic *in dark* level of the capacitor stages within the allowed thresholds.

The dynamics of two coupled LCOs can be represented in a two-dimensional torus, T^2. To be specific, according to the stage of each LCO there are four sections on T^2, each having a different timescale ratio. Approximate conclusions about orbit commensurability can be drawn if the LCOs are near the singular limit. In our domain of parameters, states where all LCOs are firing have an almost null probability of occurring. Closed orbits or dense ones are only a consequence of the ratio between charging characteristic timescales in the same way as any analytical flux in T^2.

In order to study quantitatively the dynamics of coupled LCOs, two different Phase Point Transformations (PPT) can be employed. Both PPT are based on the Poincaré sections formed by the upper and lower thresholds of the leading LCO. Linear PPT only uses one of them, thus only the period evolution is solved, while the nonlinear PPT uses both thresholds in order to preserve the piecewise character. The linear PPT erases the two-time scales present and orbit commensurability is resolved only as a function of the ratio be-

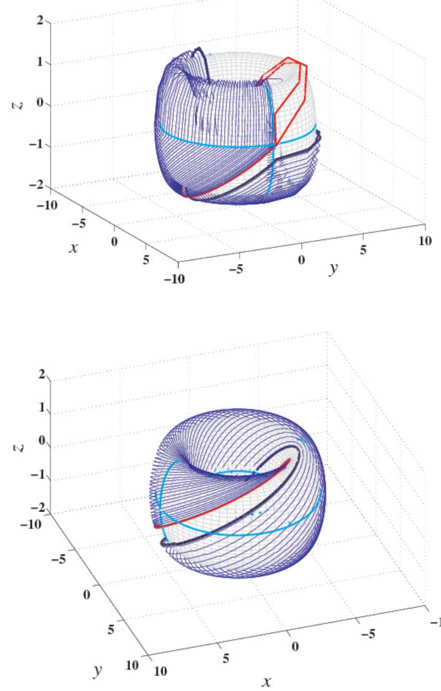

Figure 1 – Experimentally nonlinear (top) and linear (bottom) PPT for a pair of coupled LCOs on the $(1:1)$ synchronization state and in the spiking regime. Red line corresponds to the synchronized state, the black one to the initial uncoupled state and the blue ones are the orbit transients. In gray, a torus is shown with cyan lines that divide each sector.

tween LCO periods ($T_1/T_2 \simeq m/n$ for a $(n:m)$ relation), while the nonlinear one takes into account the piecewise property of the flux as shown in the top graph of Fig. 1. The state where both LCOs are discharging is badly represented due to the 10 kHz sampling rate used.

It can be seen, Fig. 1 (bottom), that the linear PPT exhibits in a better way that the initial state and the final orbit are non-intersecting trajectories on T^2. Besides, the synchronized state constitutes an attracting limit cycle of the coupled system. In Ref. [4] is shown that for different initial conditions synchronization times vary, though an attracting limit cycle exists, as well as a repelor that is only reachable for a single initial condition.

There are several locking possibilities according to the frequency ratio of the oscillators. Whenever this ratio is near a rational number, synchronization can be achieved. This condition is represented by the phase difference between oscillators $|n\Phi_i - m\Phi_j| < \epsilon_{(n:m)}$. Figure 2 shows experimental and numerical results for several synchronization states as a function of in dark period detuning. Results corresponding to MI configuration exhibit symmetry when detuning is accomplished by winding up or down LCO_2 period, while in MS configuration (not shown here) the tongues are tilted. Certain analytical calculations allow to obtain functional relations for the stability regions using T_λ as a control parameter. Moreover, the first return map is able to show stability conditions for the synchronization edges.

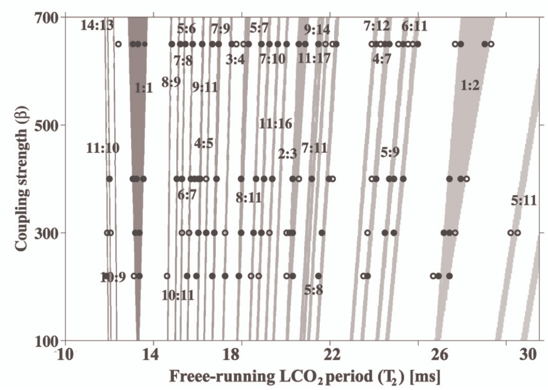

Figure 2 – MI synchronization regions achieved by decreasing the LCO_2 period while LCO_1, was kept constant at $T_1^{dark} = 13.3$ ms. Black circles correspond to experimental results. Filled ones, are synchronized states, and circles define the transition state where phase-slips are observed. Labels signal the integer ratios. The winding number is: $\rho = T_1/T_2$.

To summarize, we studied the dynamics of two coupled LCOs. PPT used to represent the dynamics in T^2 showed that simple linear transformations constructed from a single Poincaré section erase the piecewise character of the flux. Thus, a piecewise linear alternative is presented formed by the use of two Poincaré sections which correspond to divide S^1 in two sections. Resultant commensurability issues related to having four different sections in T^2 enrich the possible dynamical considerations regarding dense or closed orbits. This new phase transformation is usable in any piecewise system as can be easily generalized if the dynamics exhibit more than one characteristic timescale. Finally, physical models of biological systems, such as the one dealt here, provide insight into fundamental mechanisms of those systems.

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